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XLII. *A Method of extending Cardan's Rule for resolving one Case of a Cubick Equation of this Form, $x^3 - qx = r$, to the other Case of the same Equation, which it is not naturally fitted to solve, and which is therefore often called the irreducible Case. By Francis Maferes, Esq. F. R. S. Cursitor Baron of the Exchequer.*

Read July 9, 1778.

A R T I C L E I.

IT is well known to all persons conversant with algebra, that CARDAN's rule for resolving the cubick equation $x^3 - qx = r$ is only fitted to resolve it when $\frac{r^2}{4}$ is equal to, or greater than, $\frac{q^3}{27}$, or when r is equal to, or greater than, $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and that it is of no use in the resolution of the other case of this equation, in which r is of any magnitude less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$. For in this case $\frac{r^2}{4} - \frac{q^3}{27}$ becomes (according to the usual language of algebraists) a negative quantity, and consequently its square-root be-

comes impossible, and the expression given by CARDAN'S rule for the value of x (which is either $\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$

$$\frac{+ q}{3\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}} \text{ or } \sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}},$$

involves in it the impossible quantity $\sqrt{\frac{rr}{4} - \frac{q^3}{27}}$, and therefore is unintelligible and useless: or, according to what appears to me a more correct way of speaking (who never could form any idea of a negative quantity, and never understand by the sign — any thing more than the subtraction of a lesser quantity from a greater), the quantity $\frac{rr}{4} - \frac{q^3}{27}$ becomes itself impossible, or the supposition that $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, (which is one of the foundations of CARDAN'S rule), is no longer true, and consequently the rule itself, which is built upon it, can no longer take place.

2. Nevertheless it is possible, by the help of Sir ISAAC NEWTON's binomial theorem, to extend this rule to this latter case, in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, and which it is not of itself fitted to resolve; or, to speak with more accuracy, it is possible to derive from the expression of the value of x given by CARDAN'S rule for the resolution of the equation $x^3 - qx = r$ in the first case, in which $\frac{rr}{4}$ is greater

greater than $\frac{q^3}{27}$, another expression somewhat different from the former, that shall exhibit the true value of x in the second case, in which $\frac{r}{4}$ is less than $\frac{q^3}{27}$, provided it be not less than $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$: and this without any mention of either impossible, or negative, quantities. To shew how this may be effected, is the design of the following pages.

3. That the whole of this matter may be seen at one view, it will be convenient to set forth the foundation and investigation of CARDAN's rule for resolving the equation $x^3 - qx = r$, when $\frac{r}{4}$ is greater than $\frac{q^3}{27}$; which may be done as follows.

*The Investigation of Cardan's Rule for resolving the Cubick
Equation $x^3 - qx = r$, when $\frac{r}{4}$ is greater than $\frac{q^3}{27}$.*

4. Previously to the investigation of this rule, it will be proper to make the following observations.

OBS. 1. In the equation $x^3 - qx = r$ (which is a proposition affirming that x^3 is greater than qx , and that the excess is equal to r) xx must always be greater than q , and x than $\sqrt[3]{q}$.

OBS.

OBS. 2. While x increases from \sqrt{q} ad infinitum, x^3 will increase continually from $q\sqrt{q}$ ad infinitum, and qx will increase continually from the same quantity $q\sqrt{q}$ ad infinitum.

OBS. 3. Also, while x increases from \sqrt{q} ad infinitum, the excess of x^3 above qx will increase continually from nothing ad infinitum, without ever decreasing. For, if we put \dot{x} to denote the increment which x receives in any given time, either small or great, $q\dot{x}$ will be the increment which qx will receive in the same time, and $3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$ will be the increment of x^3 in the same time. Now, since xx is always greater than q during the whole increase of x from being equal to \sqrt{q} ad infinitum, $xx \times \dot{x}$ will be greater than $q\dot{x}$ during that whole increase. Therefore, *à fortiori*, $3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$ (which is more than triple of $xx \times \dot{x}$) will be greater than $q\dot{x}$; that is, the increment of x^3 will be greater than the contemporary increment of qx during all the increase of x . Consequently the excess of x^3 above qx , or the compound quantity $x^3 - qx$, will continually increase, without ever decreasing, while x increases from \sqrt{q} to any greater magnitude.

OBS. 4. Since the compound quantity $x^3 - qx$ increases continually at the same time as x increases; and, when

x is equal to $\frac{2\sqrt{q}}{\sqrt{3}}$, $x^3 - qx$ is $(=\frac{8q\sqrt{q}}{3\sqrt{3}} - \frac{2q\sqrt{q}}{\sqrt{3}} = \frac{8q\sqrt{q}}{3\sqrt{3}} - \frac{6q\sqrt{q}}{3\sqrt{3}}) = \frac{2q\sqrt{q}}{3\sqrt{3}}$, it follows that, if x is greater than $\frac{2\sqrt{q}}{\sqrt{3}}$, the compound quantity $x^3 - qx$ will be greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and, if x is less than $\frac{2\sqrt{q}}{\sqrt{3}}$, the said compound quantity will be less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$; and, è *converso*, if the compound quantity $x^3 - qx$, or, its equal, the absolute term r , is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the value of x will be greater than $\frac{2\sqrt{q}}{\sqrt{3}}$; and, if $x^3 - qx$, or r , is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the value of x will be less than $\frac{2\sqrt{q}}{\sqrt{3}}$; or, if $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, x will be greater than $\frac{2\sqrt{q}}{\sqrt{3}}$, and, if $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, x will be less than $\frac{2\sqrt{q}}{\sqrt{3}}$.

OBS. 5. When r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, and consequently (by the last observation) x is greater than $\frac{2\sqrt{q}}{\sqrt{3}}$, xx will be greater than $\frac{4q}{3}$, and $\frac{xx}{4}$ will be greater than $\frac{q}{3}$. But $\frac{xx}{4}$ is the square of $\frac{x}{2}$. Therefore when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, the square of half x will be greater than $\frac{q}{3}$. But (by EUCLID's Elements, Book II. Prop. V.) it is always possible to divide a line, as x , into two unequal parts in such

a pro-

a proportion that the rectangle under its parts shall be equal to any quantity that is less than the square of its half. Therefore, when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, it is possible to divide the line, or root, x into two unequal parts of such magnitudes that their rectangle, or product, shall be equal to $\frac{q}{3}$. This observation is the foundation of CARDAN's rule for the resolution of the equation $x^3 - qx = r$ in the first case of that equation, or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$; the investigation of which is as follows.

P R O B L E M.

5. To resolve the Equation $x^3 - qx = r$, when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$.

S O L U T I O N.

Since r is supposed to be greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and consequently (by Obs. 5.) $\frac{xx}{4}$ is greater than $\frac{q}{3}$, it is possible for x to be divided into two unequal parts of such magnitudes that their rectangle, or product, shall be equal to $\frac{q}{3}$. Let it be conceived to be so divided; and let the

greater of the two parts be called a , and the lesser b . Then will ab be $= \frac{q}{3}$, and consequently $3ab$ will be $= q$, and $3ab \times \sqrt{a+b}$ will be $= q \times \sqrt{a+b}$.

Now, since $a+b$ is equal to x , we shall have $a^3 + 3aab + 3abb + b^3 = x^3$, and $q \times \sqrt{a+b} = qx$. Therefore $x^3 - qx$ will be $= a^3 + 3aab + 3abb + b^3 - q \times \sqrt{a+b} = a^3 + 3ab \times \sqrt{a+b} + b^3 - q \times \sqrt{a+b}$; that is (because $3ab \times \sqrt{a+b}$ is $= q \times \sqrt{a+b}$) $x^3 - qx$ will be $= a^3 + b^3$. Therefore r (which is $= x^3 - qx$) will be $= a^3 + b^3$.

But, since $3ab$ is $= q$, we shall have $b = \frac{q}{3a}$, and $b^3 = \frac{q^3}{27a^3}$. Therefore $a^3 + b^3$ is $= a^3 + \frac{q^3}{27a^3}$, and r (which is $= a^3 + b^3$) is $= a^3 + \frac{q^3}{27a^3}$. Therefore ra^3 is $= a^6 + \frac{q^3}{27}$, and $ra^3 - a^6$ is $= \frac{q^3}{27}$.

But $ra^3 - a^6$ is the product of the multiplication of $r - a^3$ into a^3 , which are together equal to r . Therefore (by El. II. 5.) $ra^3 - a^6$ must be less than the square of half r , that is, than $\frac{r}{4}$, and consequently may be subtracted from it. Let it, and its equal $\frac{q^3}{27}$, be so subtracted. And we shall have $\frac{r}{4} - ra^3 + a^6 = \frac{r}{4} - \frac{q^3}{27}$. Therefore the square-root of $\frac{r}{4} - ra^3 + a^6$ will be equal to $\sqrt{\frac{r}{4} - \frac{q^3}{27}}$. But the square-root of $\frac{r}{4} - ra^3 + a^6$ is the difference of $\frac{r}{2}$ and a^3 , that

that is, either $\frac{r}{2} - a^3$ or $a^3 - \frac{r}{2}$, according as $\frac{r}{2}$ or a^3 is the greater quantity. But it has appeared above that a^3 and b^3 together are equal to r ; and a is supposed to be greater than b , and consequently a^3 is greater than b^3 . Therefore a^3 must be greater, and b^3 less, than $\frac{r}{2}$. Therefore $a^3 - \frac{r}{2}$ is the difference of a^3 and $\frac{r}{2}$, and consequently is the square-root of the quantity $\frac{rr}{4} - ra^3 + a^6$. Therefore $a^3 - \frac{r}{2}$ is $= \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$, and a^3 is $= \frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. Consequently a is $= \sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$. But b has been shewn to be $= \frac{q}{3a}$. Therefore b is $= -\frac{q}{3\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}$; and consequently $a+b$, or x , is $= \sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + -\frac{q}{3\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}$.

Q. E. I.

6. This expression may be rendered more simple by substituting the single letter s in it instead of $\sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. For then it will be $\sqrt[3]{\frac{r}{2} + s} + \frac{q}{3\sqrt[3]{\frac{r}{2} + s}}$.

Synthetick Demonstration of the Truth of the foregoing Solution.

7. That this expression is equal to x in the equation

$x^3 - qx = r$ will appear by substituting it instead of x in the compound quantity $x^3 - qx$, which will thereby be seen to be equal to r , as it ought to be.

This may be done in the manner following.

$$\text{Since } x \text{ is } \sqrt[3]{\frac{r}{2} + s + \frac{q}{3\sqrt[3]{\frac{r}{2} + s}}}, \text{ or } \left(\frac{r}{2} + s\right)^{\frac{1}{3}} + \frac{q}{3\sqrt[3]{\frac{r}{2} + s}},$$

$$\text{we shall have } x^3 = \frac{r}{2} + s + 3 \times \left(\frac{r}{2} + s\right)^{\frac{2}{3}} \times \frac{q}{3 \times \sqrt[3]{\frac{r}{2} + s}} + 3 \times \left(\frac{r}{2} + s\right)^{\frac{1}{3}}$$

$$\times \frac{qq}{9 \times \sqrt[3]{\frac{r}{2} + s}^{\frac{2}{3}}} + \frac{q^3}{27 \times \sqrt[3]{\frac{r}{2} + s}} = \frac{r}{2} + s + q \times \left(\frac{r}{2} + s\right)^{\frac{1}{3}} + \frac{qq}{3 \times \sqrt[3]{\frac{r}{2} + s}^{\frac{1}{3}}} +$$

$$\frac{q^3}{27r + 27s}; \text{ and } qx = q \times \left(\frac{r}{2} + s\right)^{\frac{1}{3}} + \frac{qq}{3 \times \sqrt[3]{\frac{r}{2} + s}^{\frac{1}{3}}}; \text{ and conse-}$$

$$\text{quently } x^3 - qx = \frac{r}{2} + s + \frac{q^3}{27r + 27s} = \frac{r}{2} + s + \frac{q^3}{27r + 54s} = \frac{r}{2} + s +$$

$$\frac{q^3}{27r + 54s} = \frac{r}{2} + s + \frac{2q^3}{27r + 54s}.$$

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$$\text{Now } ss, \text{ or } \frac{rr}{4} - \frac{q^3}{27}, \text{ is } \frac{27rr - 4q^3}{108} = \frac{27rr - 4q^3}{36 \times 3}; \text{ or, if we put } mm = 27rr - 4q^3, \text{ we shall have } ss, \text{ or } \frac{rr}{4} - \frac{q^3}{27} = \frac{mm}{36 \times 3}, \text{ and } s = \frac{m}{6\sqrt{3}}. \text{ Therefore } \frac{2q^3}{27r + 54s} \text{ is } \frac{2q^3}{27r + 54 \times \frac{m}{6\sqrt{3}}} = \frac{2q^3}{6 \times 27 \times \sqrt{3} \times r + 54m} =$$

$$\frac{12\sqrt{3} \times q^3}{6 \times 27 \sqrt{3} \times r + 54m} = \frac{2\sqrt{3} \times q^3}{27 \sqrt{3} \times r + 9m}. \text{ Therefore } s + \frac{2q^3}{27r + 54s} \text{ is } = \frac{m}{6\sqrt{3}} +$$

$$\frac{2\sqrt{3} \times q^3}{27 \sqrt{3} \times r + 9m} = \frac{27\sqrt{3} \times rm + 9mm + 36q^3}{6 \times 27 \times 3r + 6 \times 9 \times \sqrt{3} \times m} = \frac{3\sqrt{3} \times rm + mm + 4q^3}{54r + 6\sqrt{3} \times m} =$$

$$\frac{3\sqrt{3} \times rm + 27rr - 4q^3 + 4q^3}{54r + 6\sqrt{3} \times m} = \frac{3\sqrt{3} \times rm + 27rr}{54r + 6\sqrt{3} \times m} = \frac{\sqrt{3} \times rm + 9rr}{18r + 2\sqrt{3} \times m}; \text{ and } \frac{r}{2} +$$

 $s +$

Second Case of the Cubick Equation $x^3 - qx = r$. 911

$s + \frac{2q^3}{27r+54}$ is $= \frac{r}{2} + \frac{\sqrt{3} \times rm + 9rr}{18r+2\sqrt{3} \times m} = \frac{18rr+2\sqrt{3} \times rm+2\sqrt{3} \times rm+18rr}{36r+4\sqrt{3} \times m} =$
 $\frac{36rr+4\sqrt{3} \times rm}{36r+4\sqrt{3} \times m} = r$. But we have before shewn that $x^3 - qx$ is $= \frac{r}{2} + s + \frac{2q^3}{27r+54}$. Therefore $x^3 - qx$ is $= r$, and consequently $\sqrt[3]{\frac{r}{2} + s + \frac{q}{3\sqrt[3]{\frac{r}{2} + s}}}$ is the true value of x in the cubick equation $x^3 - qx = r$. Q. E. D.

Two other Expressions for the Root of the foregoing Equation.

8. Two other expressions may be found for the root of this equation by resuming the investigation contained in Art. 5. The first of these expressions is

$$\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \frac{q}{3\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}, \text{ or (if we put } ss, \text{ as}$$

$$\text{before, } = \frac{rr}{4} - \frac{q^3}{27}), \sqrt[3]{\frac{r}{2} - s + \frac{q}{3\sqrt[3]{\frac{r}{2} - s}}}. \text{ The other ex-}$$

pression is $\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}, \text{ or } \sqrt[3]{\frac{r}{2} + s + \sqrt[3]{\frac{r}{2} - s}}$. These expressions are to be found in the following manner.

Investigation of the said Expressions.

9. In Art. 5. we supposed the line x to be divided into

to two unequal parts a and b , of which a was supposed to be the greater; and we first found the value of the greater part a , and then determined that of the lesser part b from its relation to a , which is expressed by the equation $3ab = q$. But we may with the same ease first determine the value of the lesser part b , and then derive from it that of the greater part a ; which would produce the first of the two expressions of the value of x mentioned in the last article. This may be done as follows.

Since it has been shewn in Art. 5. that r is $= a^3 + b^3$, and $3ab$ is $= q$, and consequently a is $= \frac{q}{3b}$, and a^3 to $\frac{q^3}{27b^3}$, it follows that r will be $= \frac{q^3}{27b^3} + b^3$. Therefore rb^3 is $= \frac{q^3}{27} + b^6$, and (subtracting b^6 from both sides) $rb^3 - b^6$ is $= \frac{q^3}{27}$. Therefore (subtracting both sides from $\frac{rr}{4}$, than which they are evidently less), we shall have $\frac{rr}{4} - rb^3 + b^6 = \frac{rr}{4} - \frac{q^3}{27}$. Therefore the square-root of $\frac{rr}{4} - rb^3 + b^6$ will be $= \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. But the square-root of $\frac{rr}{4} - rb^3 + b^6$ is the difference of the quantities $\frac{r}{2}$ and b^3 , that is (because b^3 is the lesser part of $a^3 + b^3$, or r , and consequently is less than the half of it, or $\frac{r}{2}$), it is $= \frac{r}{2} - b^3$. Therefore $\frac{r}{2} - b^3$ is $= \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. Therefore (adding b^3 to both sides) $\frac{r}{2}$ will be $= b^3 +$

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$\sqrt{\frac{rr}{4} - \frac{q^3}{27}}$, and (subtracting $\sqrt{\frac{rr}{4} - \frac{q^3}{27}}$ from both sides) b will be $= \frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. Therefore b is $= \sqrt[3]{\frac{r}{2}} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$, and consequently $b + a$, and a ($= \frac{q}{3b}$) is $= \frac{q}{3\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}$, and consequently $b + a$, or $a + b$, or x , is $= \sqrt[3]{\frac{r}{2}} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}} + \frac{q}{3\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}$, or (if we put $ss = \frac{rr}{4} - \frac{q^3}{27}$) $\sqrt[3]{\frac{r}{2} - s} + \frac{q}{3\sqrt[3]{\frac{r}{2} - s}}$. Q. E. I.

Synthetic demonstration of the truth of the foregoing expression.

10. Here again we may demonstrate synthetically, that this expression is equal to the true value of x in the proposed equation $x^3 - qx = r$, by substituting it for x in the left-hand side of that equation. For, if we make that substitution, we shall find that the value of $x^3 - qx$ thence arising will be equal to r . This may be done in the manner following.

If x is $= \sqrt[3]{\frac{r}{2} - s} + \frac{q}{3\sqrt[3]{\frac{r}{2} - s}}$, or $\sqrt[3]{\frac{r}{2} - s} + \frac{q}{3 \times \sqrt[3]{\frac{r}{2} - s}}$, we shall have $x^3 = \frac{r}{2} - s + 3 \times \left[\sqrt[3]{\frac{r}{2} - s}\right]^2 \times \frac{q}{3 \times \sqrt[3]{\frac{r}{2} - s}} + 3 \times \left[\sqrt[3]{\frac{r}{2} - s}\right] \times$

$$\begin{aligned}
 & \frac{q\bar{q}}{9 \times \sqrt[3]{\frac{r}{2} - s}} + \frac{q^3}{27 \times \sqrt[3]{\frac{r}{2} - s}} = \frac{r}{2} - s + q \times \sqrt{\frac{r}{2} - s}^{\frac{1}{3}} + \frac{q\bar{q}}{3 \times \sqrt[3]{\frac{r}{2} - s}} + \\
 & \frac{q^3}{27 \times \sqrt[3]{\frac{r}{2} - s}}, \text{ and } qx = q \times \sqrt{\frac{r}{2} - s}^{\frac{1}{3}} + \frac{q\bar{q}}{3 \times \sqrt[3]{\frac{r}{2} - s}}, \text{ and conse-} \\
 & \text{quently } x^3 - qx = \frac{r}{2} - s + \frac{q^3}{27 \times \sqrt[3]{\frac{r}{2} - s}} = \frac{r}{2} - s + \frac{q^3}{\frac{27r}{2} - 27s} = \frac{r}{2} - s \\
 & + \frac{q^3}{\frac{27r}{2} - \frac{54s}{2}} = \frac{r}{2} - s + \frac{2q^3}{27r - 54s}. \quad \text{Now } ss, \text{ or } \frac{rr}{4} - \frac{q^3}{27} \text{ is } = \frac{27rr - 4q^3}{108} = \\
 & \frac{27rr - 4q^3}{36 \times 3}. \quad \text{Therefore if we put } mm = 27rr - 4q^3, \text{ we shall} \\
 & \text{have } ss = \frac{mm}{36 \times 3}, \text{ and } s = \frac{m}{6\sqrt{3}}. \quad \text{Therefore } \frac{2q^3}{27r - 54s} \text{ is } = \\
 & \frac{2q^3}{27r - \frac{54m}{6\sqrt{3}}} = \frac{2q^3}{6 \times 27 \times \sqrt{3} \times r - 54m} = \frac{12\sqrt{3} \times q^3}{6 \times 27 \times \sqrt{3} \times r - 54m} = \frac{2\sqrt{3} \times q^3}{27\sqrt{3} \times r - 9m}. \\
 & \text{Therefore } \frac{r}{2} - s + \frac{2q^3}{27r - 54s} \text{ is } = \frac{r}{2} - \frac{m}{6\sqrt{3}} + \frac{2\sqrt{3} \times q^3}{27 \times \sqrt{3} \times r - 9m} = \frac{r}{2} \\
 & - \frac{27 \times \sqrt{3} \times rm + 9mm + 36q^3}{6 \times 27 \times 3r - 54\sqrt{3} \times m} = \frac{r}{2} - \frac{3\sqrt{3} \times rm + mm + 4q^3}{54r - 6\sqrt{3} \times m} = \frac{r}{2} \\
 & - \frac{3\sqrt{3} \times rm + 27rr - 4q^3 + 4q^3}{54r - 6\sqrt{3} \times m} = \frac{r}{2} - \frac{3\sqrt{3} \times rm + 27rr - 54rr - 6\sqrt{3} \times rm - 6\sqrt{3} \times rm + 54rr}{54r - 6\sqrt{3} \times m} \\
 & = \frac{108rr - 12\sqrt{3} \times rm}{108r - 12\sqrt{3} \times m} = r. \quad \text{But it has been before shewn that} \\
 & x^3 - qx \text{ is } = \frac{r}{2} - s + \frac{2q^3}{27r - 54s}. \quad \text{Therefore } x^3 - qx \text{ is } = r; \text{ and} \\
 & \sqrt[3]{\frac{r}{2} - s + \frac{q}{3\sqrt[3]{\frac{r}{2} - s}}} \text{ is the true value of } x \text{ in the cubick}
 \end{aligned}$$

equation $x^3 - qx = r$. Q. E. D.

Investigation of the third expression of the value of the root x .

11. The third expression for the value of x , or the last of the two mentioned in Art. 8. to wit, $\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$, or $\sqrt[3]{\frac{r}{2} + s} + \sqrt[3]{\frac{r}{2} - s}$, may be obtained as follows.

Since $a^3 + b^3$ is $= r$, it follows that b^3 will be $= r - a^3$. But a^3 is shewn in Art. 5. to be $= \frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. Therefore $r - a^3$ is $= r - \left(\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}\right) = \frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. Consequently b^3 is $= \frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$, and b is $= \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$, and $a+b$, or x , is $= \sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$, or (putting ss , as before, $= \frac{rr}{4} - \frac{q^3}{27}$), $\sqrt[3]{\frac{r}{2} + s} + \sqrt[3]{\frac{r}{2} - s}$. Q. E. I.

Synthetic demonstration of the truth of the said third expression.

12. Here again we may demonstrate synthetically, that this expression is equal to the true value of x in the equation $x^3 - qx = r$, by substituting it for x in the left-hand side of the said equation. For, if we make that substitution, we shall find that the value of $x^3 - qx$,

thence arising, will be equal to r . This may be done in the following manner.

If x is $\sqrt[3]{\frac{r}{2} + s} + \sqrt[3]{\frac{r}{2} - s}$, or $\left[\frac{r}{2} + s\right]^{\frac{1}{3}} + \left[\frac{r}{2} - s\right]^{\frac{1}{3}}$, we shall have $x^3 = \frac{r}{2} + s + 3 \times \left[\frac{r}{2} + s\right]^{\frac{2}{3}} \times \left[\frac{r}{2} - s\right]^{\frac{1}{3}} + 3 \times \left[\frac{r}{2} + s\right]^{\frac{1}{3}} \times \left[\frac{r}{2} - s\right]^{\frac{2}{3}}$
 $+ \frac{r}{2} - s = r + 3 \times \left[\frac{r}{2} + s\right]^{\frac{2}{3}} \times \left[\frac{r}{2} - s\right]^{\frac{1}{3}} + 3 \times \left[\frac{r}{2} + s\right]^{\frac{1}{3}} \times \left[\frac{r}{2} - s\right]^{\frac{2}{3}} = r +$
 $3 \times \left[\frac{r}{2} + s\right]^{\frac{1}{3}} \times \left[\frac{r}{2} + s\right]^{\frac{1}{3}} \times \left[\frac{r}{2} - s\right]^{\frac{1}{3}} + 3 \times \left[\frac{rr}{4} + ss\right]^{\frac{1}{3}} \times \left[\frac{r}{2} - s\right]^{\frac{1}{3}} \times \left[\frac{r}{2} - s\right]^{\frac{1}{3}} =$
 $r + 3 \times \left[\frac{r}{2} + s\right]^{\frac{1}{3}} \times \left[\frac{rr}{4} + ss\right]^{\frac{1}{3}} + 3 \times \left[\frac{rr}{4} + ss\right]^{\frac{1}{3}} \times \left[\frac{r}{2} - s\right]^{\frac{1}{3}} = r + 3 \times \left[\frac{r}{2} + s\right]^{\frac{1}{3}} \times$
 $\times \left[\frac{rr}{4} - \frac{rr}{4} + \frac{q^3}{27}\right]^{\frac{1}{3}} + 3 \times \left[\frac{rr}{4} - \frac{rr}{4} + \frac{q^3}{27}\right]^{\frac{1}{3}} \times \left[\frac{r}{2} - s\right]^{\frac{1}{3}} = r + 3 \times \left[\frac{r}{2} + s\right]^{\frac{1}{3}} \times \left[\frac{q^3}{27}\right]^{\frac{1}{3}} +$
 $3 \times \left[\frac{q^3}{27}\right]^{\frac{1}{3}} \times \left[\frac{r}{2} - s\right]^{\frac{1}{3}} = r + 3 \times \left[\frac{r}{2} + s\right]^{\frac{1}{3}} \times \frac{q}{3} + 3 \times \frac{q}{3} \times \left[\frac{r}{2} - s\right]^{\frac{1}{3}} = r + q \times$
 $\left[\frac{r}{2} + s\right]^{\frac{1}{3}} + q \times \left[\frac{r}{2} - s\right]^{\frac{1}{3}}.$ And qx will be $= q \times \left[\frac{r}{2} + s\right]^{\frac{1}{3}} + q \times$
 $\left[\frac{r}{2} - s\right]^{\frac{1}{3}}.$ Therefore $x^3 - qx$ will be $= r + q \times \left[\frac{r}{2} + s\right]^{\frac{1}{3}} + q \times$
 $\left[\frac{r}{2} - s\right]^{\frac{1}{3}} - q \times \left[\frac{r}{2} + s\right]^{\frac{1}{3}} - q \times \left[\frac{r}{2} - s\right]^{\frac{1}{3}} = r;$ and consequently
 $\left[\frac{r}{2} + s\right]^{\frac{1}{3}} + \left[\frac{r}{2} - s\right]^{\frac{1}{3}},$ or $\sqrt[3]{\frac{r}{2} + s} + \sqrt[3]{\frac{r}{2} - s},$ is the true value of x in the cubick equation $x^3 - qx = r.$

13. N.B. I do not remember to have seen these substitutions, or synthetical demonstrations of the truth of the expressions given by CARDAN's rule, in any book of algebra.

14. An Example of the Resolution of a Cubick Equation of the aforesaid Form, $x^3 - qx = r$, by means of each of the three foregoing Expressions.

I will here insert a single example of a numeral equation of the foregoing form, $x^3 - qx = r$, resolved by each of the three expressions above-mentioned, in order to shew that they will all three bring out the same number for its root..

Let it therefore be required to find the value of x in the cubick equation $x^3 - 3x = 18$.

15. In this equation q is = 3, and r is = 18. Therefore $\sqrt[3]{q}$ is = $\sqrt{3}$, and $\frac{\sqrt[3]{q}}{3\sqrt[3]{3}}$ is = $\frac{\sqrt[3]{2 \times 3}}{3\sqrt[3]{3}} = 2$, which is greatly less than 18, or r . Therefore this equation comes under the above-mentioned rule, and may be resolved by either of the foregoing expressions.

Resolution of the equation $x^3 - 3x = 18$ by the first of the said expressions.

16. The first of those expressions is $\sqrt[3]{\frac{r}{2} + s} + \frac{q}{3\sqrt[3]{\frac{r}{2} + s}}$, in which s stands for $\sqrt{\frac{rr}{4} - \frac{q^2}{27}}$.

Now

Now, since q is = 3, $\frac{q}{3}$ will be = $\frac{3}{3} = 1$, and consequently $\frac{q^3}{27}$, or the cube of $\frac{q}{3}$, will also be = 1. And, since r is = 18, we shall have $\frac{r}{2} = 9$, and $\frac{rr}{4} = 81$, and consequently $\frac{rr}{4} - \frac{q^3}{27} = 81 - 1 = 80$; that is, ss will be = 80. Therefore s is = $\sqrt{80} = \sqrt{16 \times \sqrt{5}} = 4\sqrt{5}$; and $\frac{r}{2} + s$ is = $9 + 4\sqrt{5} = \frac{72 + 32\sqrt{5}}{8} = \frac{27 + 27\sqrt{5} + 45 + 5\sqrt{5}}{8}$; and consequently $\sqrt[3]{\frac{r}{2} + s}$ is = $\frac{3 + \sqrt{5}}{2}$. Therefore $3 \times \sqrt[3]{\frac{r}{2} + s}$ is = $\frac{3 \times [3 + \sqrt{5}]}{2}$, and $\frac{q}{3\sqrt[3]{\frac{r}{2} + s}}$ is = $3 \times \frac{2}{3 \times 3 + \sqrt{5}} = \frac{2}{3 + \sqrt{5}}$; and $\sqrt[3]{\frac{r}{2} + s} + \frac{q}{3\sqrt[3]{\frac{r}{2} + s}}$ is = $\frac{3 + \sqrt{5}}{2} + \frac{2}{3 + \sqrt{5}} = \frac{(3 + \sqrt{5}) \times (3 + \sqrt{5}) + 4}{2 \times 3 + \sqrt{5}} = \frac{9 + 6\sqrt{5} + 5 + 4}{2 \times 3 + \sqrt{5}} = \frac{18 + 6\sqrt{5}}{2 \times 3 + \sqrt{5}} = \frac{6 \times 3 + \sqrt{5}}{2 \times 3 + \sqrt{5}} = \frac{6}{2} = 3$. Therefore 3 is the value of x in the equation $x^3 - 3x = 18$. And so we shall find it to be upon trial: for, if x is taken = 3, we shall have $x^3 = 27$, and $3x = 3 \times 3 = 9$, and $x^3 - 3x = 27 - 9 = 18$. And thus we see that the first of the three foregoing expressions, to wit, $\sqrt[3]{\frac{r}{2} + s} + \frac{q}{3\sqrt[3]{\frac{r}{2} + s}}$, has given us the true value of x in this equation.

Resolution of the same equation by the second and third of the foregoing expressions.

17. We are now to resolve the same equation $x^3 - 3x = 18$ by means of the two other expressions, to wit,

$$\sqrt[3]{\frac{r}{2} - s} + \frac{q}{3\sqrt[3]{\frac{r}{2} - s}}, \text{ and } \sqrt[3]{\frac{r}{2} + s} + \sqrt[3]{\frac{r}{2} - s}.$$

Now, since r is = 18, and s has been shewn to be $= \sqrt{80}$, or $4\sqrt{5}$, we shall have $\frac{r}{2} - s = 9 - 4\sqrt{5} = \frac{72 - 32\sqrt{5}}{8} = \frac{27 - 27\sqrt{5} + 45 - 5\sqrt{5}}{8}$, and $\sqrt[3]{\frac{r}{2} - s} = \frac{3 - \sqrt{5}}{2}$. Therefore

$$3\sqrt[3]{\frac{r}{2} - s} \text{ is } \frac{3 \times 3 - \sqrt{5}}{2}, \text{ and } \frac{q}{3\sqrt[3]{\frac{r}{2} - s}} \text{ is } = \frac{\frac{3}{2}}{\frac{3 \times 3 - \sqrt{5}}{2}} = 3 \times$$

$$\frac{\frac{2}{3}}{\frac{3 - \sqrt{5}}{2}} = \frac{2}{3 - \sqrt{5}}. \text{ Consequently } \sqrt[3]{\frac{r}{2} - s} + \frac{q}{3\sqrt[3]{\frac{r}{2} - s}} \text{ is } =$$

$$\frac{3 - \sqrt{5}}{2} + \frac{2}{3 - \sqrt{5}} = \frac{3 - \sqrt{5} \times 3 - \sqrt{5} + 4}{2 \times 3 - \sqrt{5}} = \frac{9 - 6\sqrt{5} + 5 + 4}{2 \times 3 - \sqrt{5}} = \frac{18 - 6\sqrt{5}}{2 \times 3 - \sqrt{5}} = \frac{9 - 3\sqrt{5}}{3 - \sqrt{5}} =$$

$$\frac{3 \times 3 - \sqrt{5}}{3 - \sqrt{5}} = 3. \text{ Therefore } x \text{ is } = 3, \text{ as it was found to be}$$

by the first expression.

18. The third expression $\sqrt[3]{\frac{r}{2} + s} + \sqrt[3]{\frac{r}{2} - s}$ is in the present case $= \frac{3 + \sqrt{5}}{2} + \frac{3 - \sqrt{5}}{2} = \frac{6}{2} = 3$. Therefore by this expression, as well as by both the former, the value of x in the equation $x^3 - 3x = 18$ comes out to be 3.

19. Note. The foregoing method of resolving the cubick equation $x^3 - qx = r$, when r is greater than $\frac{2q\sqrt[3]{q}}{3\sqrt[3]{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, and a like method of resolving the cubick equation $x^3 + qx = r$ (which holds good in all cases, whatever be the magnitudes of q and r), are usually known by the name of CARDAN's rules, because they were first published by him in his treatise of algebra, intitled, *Ars magna, quam vulgo Cossam vocant, seu regulas Algebraicas*, in the year 1545, although, as he himself informs us, they were first found out by one SCIPIO FERREUS of Bononia. See WALLIS'S algebra, Chap. XIII.

Of the second case of the cubick equation $x - qx = r$; in which r is less than $\frac{2q\sqrt[3]{q}}{3\sqrt[3]{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, and which cannot be resolved by CARDAN's rule.

20. The remaining case of the cubick equation $x^3 - qx = r$, in which r is less than $\frac{2q\sqrt[3]{q}}{3\sqrt[3]{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, and which consequently cannot be resolved by the rules above-mentioned, has, upon that account, obtained amongst algebraists the name of the *irreducible case*: at least it is often called by the French writers of algebra *le cas irréductible*. The object of the remaining pages of

of this paper is to shew how, by the help of Sir ISAAC NEWTON's famous binomial theorem, the foregoing solution of the other, or first, case of this equation may be, as it were, extended to this latter case, or, rather, may be made the means of discovering, by a very peculiar train of reasoning, another solution, that shall be adapted to it.

21. By the binomial theorem it appears that the cube-root of the binomial quantity $a + b$ (in which a is supposed to be greater than b) is equal to the following infinite series, to wit, $a^{\frac{1}{3}} + \frac{a^{\frac{1}{3}}b}{3a} - \frac{a^{\frac{1}{3}}b^2}{9a^2} + \frac{5a^{\frac{1}{3}}b^3}{81a^3} - \frac{10a^{\frac{1}{3}}b^4}{243a^4} + \frac{22a^{\frac{1}{3}}b^5}{729a^5} - \frac{154a^{\frac{1}{3}}b^6}{6561a^6} + \frac{2618a^{\frac{1}{3}}b^7}{137,781a^7} - \&c.$, or to $a^{\frac{1}{3}} \times$ the infinite series $1 + \frac{b}{3a} - \frac{b^2}{9a^2} + \frac{5b^3}{81a^3} - \frac{10b^4}{243a^4} + \frac{22b^5}{729a^5} - \frac{154b^6}{6561a^6} + \frac{2618b^7}{137,781a^7} - \&c.$ or (if we put the capital letters A, B, C, D, E, F, G, H, &c. for the several numeral coefficients, 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{5}{81}$, $\frac{10}{243}$, $\frac{22}{729}$, $\frac{154}{6561}$, $\frac{2618}{137,781}$, &c. of the terms of the series, respectively,) $a^{\frac{1}{3}} \times$ the infinite series $1 + \frac{1Ab}{3a} - \frac{2Bb^2}{6a^2} + \frac{5Cb^3}{9a^3} - \frac{8Db^4}{12a^4} + \frac{11Eb^5}{15a^5} - \frac{14Fb^6}{18a^6} + \frac{17Gb^7}{21a^7} - \&c.$ in which series both the numerators and the denominators of the generating fractions, $\frac{2}{6}$, $\frac{5}{9}$, $\frac{8}{12}$, $\frac{11}{15}$, $\frac{14}{18}$, $\frac{17}{21}$, &c. following the second term, increase continually by 3, so that it will be easy for any one to continue the series to as many terms as he shall think proper.

22. In like manner the cube-root of the residual quantity $a - b$ is found by the same binomial theorem to

be equal to the infinite series $a^{\frac{1}{3}} - \frac{a^{\frac{1}{3}}b}{3a} - \frac{a^{\frac{1}{3}}b^2}{9a^2} - \frac{5a^{\frac{1}{3}}b^3}{81a^3} - \frac{10a^{\frac{1}{3}}b^4}{243a^4} -$
 $\frac{22a^{\frac{1}{3}}b^5}{729a^5} - \frac{154a^{\frac{1}{3}}b^6}{6561a^6} - \frac{2618a^{\frac{1}{3}}b^7}{137,781a^7} - \text{&c. or to } a^{\frac{1}{3}} \times \text{ the infinite series}$
 $I - \frac{b}{3a} - \frac{b^2}{9a^2} - \frac{5b^3}{81a^3} - \frac{10b^4}{243a^4} - \frac{22b^5}{729a^5} - \frac{154b^6}{6561a^6} - \frac{2618b^7}{137,781a^7} - \text{&c. or}$
 $a^{\frac{1}{3}} \times \text{ the infinite series } I - \frac{1Ab}{3a} - \frac{2Bb^2}{6a^2} - \frac{5Cb^3}{9a^3} - \frac{8Db^4}{12a^4} - \frac{11Eb^5}{15a^5} -$
 $\frac{14Fb^6}{18a^6} - \frac{17Gb^7}{21a^7} - \text{&c. in which series the numeral coefficients}$

of the several terms are the same as in the series that expresses the cube-root of $a+b$, but the terms which involve the odd powers of b (which in that series are marked with the sign +, or all added to the first term,) are in this latter series marked with the sign -, and are all to be subtracted from the first term, as well as the terms which involve the even powers of b , which are to be subtracted from the first term in both serieses.

P R O B L E M.

23. Let it now be required to resolve the first case of the cubick equation $x^3 - qx = r$, in which r is greater than $\frac{2q\sqrt[3]{q}}{3\sqrt[3]{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, by means of an infinite series derived from the expressions given by CARDAN'S rule.

S O L U T I O N.

We have seen in Art. II. that, if we be put $= \frac{rr}{4} - \frac{q^3}{27}$,
the value of x in this equation will be $= \sqrt[3]{\frac{r}{2} + s + \sqrt[3]{s^2 + \frac{q^3}{27}}}$

$\sqrt[3]{\frac{r}{2} - s}$. For the sake of avoiding fractions, let e be put $= \frac{r}{2}$. And we shall have $x = \sqrt[3]{e+s} + \sqrt[3]{e-s}$. But (by Art. 21.) $\sqrt[3]{e+s}$ is $= e^{\frac{1}{3}} \times$ the infinite series $1 + \frac{s}{3e} - \frac{ss}{9ee} + \frac{5s^3}{81e^3} - \frac{10s^4}{243e^4} + \frac{22s^5}{729e^5} - \frac{154s^6}{6561e^6} + \frac{2618s^7}{137781e^7} - \&c$; and, by Art. 22. $\sqrt[3]{e-s}$ is $= e^{\frac{1}{3}} \times$ the infinite series $1 - \frac{s}{3e} - \frac{ss}{9ee} - \frac{5s^3}{81e^3} - \frac{10s^4}{243e^4} - \frac{22s^5}{729e^5} - \frac{154s^6}{6561e^6} - \frac{2618s^7}{137781e^7} - \&c$. Therefore $\sqrt[3]{e+s} + \sqrt[3]{e-s}$ is equal to $e^{\frac{1}{3}} \times$ the sum of these two serieses, that is, to $e^{\frac{1}{3}} \times$ the infinite series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \&c$; and consequently the root of the equation $x^3 - qx = r$ is $= e^{\frac{1}{3}} \times$ the infinite series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \&c. ad infinitum.$ Q. E. F.

24. Note. This series must always converge, because ss , or $\frac{rr}{4} - \frac{q^3}{27}$, is always less than $\frac{rr}{4}$, or ee . And, when ss is considerably less than ee , or $\frac{rr}{4} - \frac{q^3}{27}$ is considerably less than $\frac{rr}{4}$, or $\frac{rr}{4}$ is very little greater than $\frac{q^3}{27}$, the convergency of the terms of this series will be sufficient to make it useful. But in other cases, when $\frac{rr}{4}$ is much greater than $\frac{q^3}{27}$, (as when it is triple, quadruple or quintuple of it, or still greater,) the terms of this series will converge so slowly as to render it very unfit for practice. And indeed in the most favourable cases it will, as I believe, be less convenient in practice than the expression

$\sqrt[3]{e+s} + \sqrt[3]{e-s}$, or $\sqrt[3]{\frac{r}{2}+s} + \sqrt[3]{\frac{r}{2}-s}$, from which it was derived. However, that it may appear that this series will exhibit the root of the equation $x^3 - qx = r$ truly, if we will take the necessary pains of computing it, I will here subjoin one example, and no more, of the resolution of a cubick equation of that form by means of it, having taken care to chuse such numbers for q and r as shall make $\frac{rr}{4}$ be but little greater than $\frac{q^3}{27}$, and consequently shall give us only a small number for the fraction $\frac{ss}{ee}$, by the continual multiplication of which the terms of the series are generated.

An example of the resolution of a cubick equation of the aforesaid form, $x^3 - qx = r$, in the first case of it, in which r is greater than $\frac{2q\sqrt[3]{q}}{3\sqrt[3]{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, by means of the expression $e^{\frac{1}{3}} \times$ the infinite series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \text{ &c. obtained in Art. 23.}$

25. Let it be required to resolve the equation $x^3 - 300x = 2108$ by means of the infinite series $e^{\frac{1}{3}} \times$
 $\boxed{2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \text{ &c. obtained in Art. 23.}}$ by the help of Sir ISAAC NEWTON's binomial theorem.

Here

Here q is = 300, and r is = 2108. Therefore $\frac{2\sqrt[3]{q}}{3\sqrt[3]{3}}$
 $is = \frac{2 \times 300 \times \sqrt{300}}{3 \times \sqrt{3}} = \frac{2 \times 100 \times \sqrt{300}}{\sqrt{3}} = \frac{2 \times 100 \times 10\sqrt{3}}{\sqrt{3}} = 2 \times 100 \times 10 = 2000$, which is less than 2108, or r . Therefore this equation comes under the case of CARDAN's rule, and consequently may be resolved by means of the infinite series $e^{\frac{1}{3}} \times \left[2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} \right] - \&c.$ if that series has been justly derived from the third expression of the value of x given by CARDAN's rule.

26. Now, since r is = 2108, $\frac{r}{2}$, or e , will be = 1054, and $\frac{r}{4}$, or ee , will be = 1,110,916. And, since q is = 300, $\frac{q}{3}$ will be = 100, and $\frac{q^3}{27}$, or the cube of $\frac{q}{3}$, will be = 1000,000; and consequently ss , or $\frac{r^2}{4} - \frac{q^3}{27}$, will be ($= 1,110,916 - 1000,000$) = 110,916. Therefore, the fraction $\frac{ss}{ee}$ is = $\frac{110,916}{1,110,916} = .0998$. Therefore $\frac{s^4}{e^4}$ is = $\overline{.0998}^2 = .009,950$, and $\frac{s^6}{e^6}$ is = $\overline{.0998}^3 = .000,992$; and $\frac{2ss}{9ee}$ is = $\frac{2}{9} \times .0998 = \frac{.1996}{9} = .022,177$; and $\frac{20s^4}{243e^4}$ is = $\frac{20}{243} \times .009,950 = \frac{.199,000}{243} = .000,818$; and $\frac{308s^6}{6561e^6}$ is = $\frac{308}{6561} \times .000,992 = \frac{.305,536}{6561} = .000,046$; and consequently, $\frac{2ss}{9ee} + \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6}$ is = $.022,177 + .000,818 + .000,046 = .023,041$; and $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6}$ is = $2 - .023,041 =$

=

$= 1.976,959$. But e is $= 1054$. Consequently, $e^{\frac{1}{3}}$, or $\sqrt[3]{e}$, is $= \sqrt[3]{1054} = 10.1768$. Therefore $e^{\frac{1}{3}} \times$ the series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \text{&c.}$ is $= 10.1768 \times 1.976,959 = 20.119,116$. Therefore the root of the equation $x^3 - 300x = 2108$ is $= 20.119,116$. Q. E. I.

27. This value of x is true to five places of figures, the more accurate value of it being $20.119,053$, as will easily appear by prosecuting it to three or four more places of figures by Mr. RAPHSON's method of approximation.

28. That 20.119 is very nearly equal to, but somewhat less than, the true value of x in the equation $x^3 - 300x = 2108$, will appear by substituting it instead of x in the left-hand side of that equation. For, if we take $x = 20.119$, we shall have $xx = 404.774,161$, and $x^3 = 8143.651,345,159$, and $300x = 6035.700$; and consequently, $x^3 - 300x = 8143.651,345,159 - 6035.700 = 2107.951,345,159$, which is somewhat less than 2108 , or the accurate value of $x^3 - 300x$ in the proposed equation $x^3 - 300x = 2108$. Therefore, 20.119 must be nearly equal to, but somewhat less than, the accurate value of x in that equation.

29. It appears therefore from this example, that this expression, $e^{\frac{1}{3}} \times$ the infinite series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \text{&c.}$ does truly exhibit the root of the equation

$x^3 - qx = r$ in that case of it which falls under CARDAN'S rule, or in which r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$.

36. I now proceed to consider the problem which is the principal object of this paper, which is to shew how from the series $e^{\frac{1}{3}} \times \left[2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \&c. \right]$, we may derive another series, differing from it only in the signs of some of the terms, by which the equation $x^3 - qx = r$ may be resolved in that other case of it which does not come under CARDAN'S rule, and in which r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$: and this without any mention of either impossible or negative quantities.

P R O B L E M.

To resolve, by means of an infinite series derived from the infinite series $e^{\frac{1}{3}} \times \left[2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \&c. \right]$ the second case of the cubick equation $x^3 - qx = r$, in which r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$.

S O L U T I O N.

37. We have seen that in the first case of the equation $x^3 - qx = r$, in which $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, the product

duct of $e^{\frac{r}{3}}$ into the series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \&c.$ ad infinitum, is equal to the root x . Now there are two different ways of computing this series, which (though not equally short and convenient in practice) are nevertheless equally just and true: and therefore they must both produce the same result for the value of the series. The first way of computing it is the common one, which consists of the following processes; to wit, first, to compute the quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$, as was done in the foregoing example, art. 26, where $\frac{rr}{4}$ was found to be = 1,110,916, and $\frac{q^3}{27}$ to be 1000,000; 2dly, to subtract $\frac{q^3}{27}$ from $\frac{rr}{4}$, in order to get the quantity ss , which is equal to their difference, and which in the foregoing example was 110,916; 3dly, to divide ss by ee , so as to obtain the value of the fraction $\frac{ss}{ee}$; as in the foregoing example we found the fraction $\frac{110,916}{1,110,916}$ to be = .0998; 4thly, to compute the powers of the value found for the fraction $\frac{ss}{ee}$; as in the foregoing example we computed those of .0998, and found its square to be .009,950, and its cube to be .000,992; 5thly, to multiply $\frac{ss}{ee}$, and its powers $\frac{s^4}{e^4}$, $\frac{s^6}{e^6}$, &c. into the co-efficients $\frac{2}{9}$, $\frac{20}{243}$, $\frac{308}{6561}$, &c. respectively,

respectively, as in the foregoing example we multiplied .0998 into $\frac{2}{9}$, and .009,950 into $\frac{20}{243}$, and .000,992 into $\frac{308}{6561}$, and found the products to be .022,117, .000,818, and .000,046; and 6thly, to subtract all the products so obtained from 2 the first term of the series. This is the common and the proper way of computing the series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \text{ &c.}$ when we want to make use of it in practice. But it may also be computed in another manner, which may be described as follows.

Instead of ss insert the compound quantity $\frac{rr}{4} - \frac{q^3}{27}$ itself, to which ss is equal, in all the terms of it. And it will be thereby converted into the following series, to wit, $2 - \frac{2}{9ee} \times \left[\frac{rr}{4} - \frac{q^3}{27} \right] - \frac{20}{243e^4} \times \left[\frac{rr}{4} - \frac{q^3}{27} \right]^2 - \frac{308}{6561e^6} \times \left[\frac{rr}{4} - \frac{q^3}{27} \right]^3 - \text{ &c. or (because } ee \text{ is } = \frac{rr}{4}, \text{ and consequently } e^4 = \frac{r^4}{16}, \text{ and } e^6 = \frac{r^6}{64} \text{)} 2 - \frac{8}{9rr} \times \left[\frac{rr}{4} - \frac{q^3}{27} \right] - \frac{320}{243r^4} \times \left[\frac{rr}{4} - \frac{q^3}{27} \right]^2 - \frac{19712}{6561r^6} \times \left[\frac{rr}{4} - \frac{q^3}{27} \right]^3 - \text{ &c. or } 2 - \frac{8}{9rr} \times \left[\frac{rr}{4} - \frac{q^3}{27} \right] - \frac{320}{243r^4} \times \left[\frac{r^4}{16} - \frac{2rrq^3}{4 \times 27} + \frac{q^6}{27 \times 27} \right] - \frac{19712}{6561r^6} \times \left[\frac{r^6}{64} - \frac{3r^4q^3}{16 \times 27} + \frac{3rrq^6}{4 \times 27 \times 27} - \frac{q^9}{27 \times 27 \times 27} \right] - \text{ &c. or (putting the Greek letters } \alpha, \beta, \gamma, \text{ &c. for the numeral co-efficients }$

$$\frac{8}{9}, \frac{320}{243}, \frac{19712}{6561}, \text{ &c. respectively}) 2 - \frac{\alpha}{rr} \times \left[\frac{rr}{4} - \frac{q^3}{27} \right] - \frac{\beta}{r^4} \times \left[\frac{r^4}{16} - \frac{2rrq^3}{4 \times 27} + \frac{q^6}{27 \times 27} \right] - \frac{\gamma}{r^6} \times \left[\frac{r^6}{64} - \frac{3r^4q^3}{16 \times 27} + \frac{3rrq^6}{4 \times 27 \times 27} - \frac{q^9}{27 \times 27 \times 27} \right] - \text{ &c.}$$

&c. or $2 - \frac{a}{4} + \frac{aq^3}{27rr} - \frac{6}{16} + \frac{26q^3}{4 \times 27rr} - \frac{6q^6}{27 \times 27r^4} - \frac{\gamma}{64} + \frac{3\gamma q^3}{16 \times 27rr}$
 $- \frac{3\gamma q^6}{4 \times 27 \times 27r^4} + \frac{\gamma q^9}{27 \times 27 \times 27r^6}$ &c. which consists of a much greater number of terms than the series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6}$ &c. from which it is derived, and in which many of the terms are much more complicated than in that former series. Nevertheless, since the compound quantity $\frac{rr}{4} - \frac{q^3}{27}$ is equal to ss , the insertion of it instead of ss in the terms of that former series cannot alter its real value, though it will make it much more difficult to compute. It must therefore be true of the new and complicated series $2 - \frac{a}{4} + \frac{aq^3}{27rr} - \frac{6}{16} + \frac{26q^3}{4 \times 27rr} - \frac{6q^6}{27 \times 27r^4} - \frac{\gamma}{64} + \frac{3\gamma q^3}{16 \times 27rr} - \frac{3\gamma q^6}{4 \times 27 \times 27r^4} + \frac{\gamma q^9}{27 \times 27 \times 27r^6}$ &c. as well as of the former series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6}$ &c. that, if it be multiplied into $e^{\frac{1}{3}}$, or $\sqrt[3]{e}$, or $\sqrt[3]{\frac{r}{2}}$, the series thereby produced will be equal to the value of x in the equation $x^3 - qx = r$, or that, if the said series be cubed, and also multiplied into q , and from its cube the product of its multiplication into q be subtracted, the remainder will be equal to r , or rather will approximate to the value of r , because, as the quantity substituted for x in the compound quantity $x^3 - qx$ is only a part of an infinite series

that is equal to x , it is impossible that the said remainder (which is produced by that substitution) should be accurately equal to the whole value of r .

32. By the help of this observation we may from the foregoing series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \&c.$ which, being multiplied into $e^{\frac{1}{3}}$, or the cube-root of $\frac{r}{2}$, expresses the value of x in the equation $x^3 - qx = r$, in the first case of that equation, when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, deduce another series resembling the former in the composition of its terms, but differing from it in the signs to be prefixed to some of them, that will likewise (if multiplied into $e^{\frac{1}{3}}$, or the cube-root of $\frac{r}{2}$) express the value of x in the second case of the same equation, in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, and which cannot be resolved by CARDAN's rules. This may be done as follows.

33. If in this second case of the equation $x^3 - qx = r$ we subtract $\frac{rr}{4}$ from $\frac{q^3}{27}$, and call the remainder ss (as we before put ss for the opposite difference $\frac{rr}{4} - \frac{q^3}{27}$) and then raise the powers of ss , to wit, s^4, s^6, s^8, s^{10} , and also the correspondent powers of its value $\frac{q^3}{27} - \frac{r^4}{4}$, to wit,

$$\left[\frac{q^3}{27} - \frac{rr}{4} \right]^2, \left[\frac{q^3}{27} - \frac{rr}{4} \right]^3, \left[\frac{q^3}{27} - \frac{rr}{4} \right]^4, \left[\frac{q^3}{27} - \frac{rr}{4} \right]^5, \&c. \text{ the even}$$

powers of the difference $\frac{q^3}{27} - \frac{rr}{4}$, to wit, $\sqrt{\frac{q^3}{27} - \frac{rr}{4}}^2$, $\sqrt{\frac{q^3}{27} - \frac{rr}{4}}^4$, &c. will consist of the very same terms, or the same powers, products, and multiples of the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$, and with the same signs + and - prefixed to them, as were before contained in the even powers of the opposite difference $\frac{rr}{4} - \frac{q^3}{27}$, when $\frac{rr}{4}$ was greater than $\frac{q^3}{27}$. Thus, for example, the square of $\frac{rr}{4} - \frac{q^3}{27}$ in the former case was $\frac{r^4}{16} - \frac{2rrq^3}{4 \times 27} + \frac{q^6}{27 \times 27}$; and in the present case the square of $\frac{q^3}{27} - \frac{rr}{4}$ is $\frac{q^6}{27 \times 27} - \frac{2q^3rr}{27 \times 4} + \frac{r^4}{16}$, which consists of the same terms, and with the same signs prefixed to them, as were contained in the square of $\frac{rr}{4} - \frac{q^3}{27}$, and differs from it only in the order in which the extreme terms $\frac{r^4}{16}$ and $\frac{q^6}{27 \times 27}$ are placed. And the same observation is true concerning all the other even powers of the opposite differences $\frac{rr}{4} - \frac{q^3}{27}$ and $\frac{q^3}{27} - \frac{rr}{4}$.

Also the odd powers of the difference $\frac{q^3}{27} - \frac{rr}{4}$, to wit, $\frac{q^3}{27} - \frac{rr}{4}$ itself, and $\sqrt{\frac{q^3}{27} - \frac{rr}{4}}^3$, $\sqrt{\frac{q^3}{27} - \frac{rr}{4}}^5$, &c. will consist of the same terms, or of the same powers, products, and multiples of the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$, as were contained in the same odd powers of the opposite difference

$\frac{rr}{4} - \frac{q^3}{27}$, when $\frac{rr}{4}$ was greater than $\frac{q^3}{27}$. But the signs pre-fixed to the said terms will be contrary to those which were prefixed to them in the former case. Thus, the cube of $\frac{rr}{4} - \frac{q^3}{27}$ in the former case was $\frac{r^6}{64} - \frac{3r^4q^3}{16 \times 27} + \frac{3r^2q^6}{4 \times 27 \times 27}$ $- \frac{q^9}{27 \times 27 \times 27}$; and the cube of $\frac{q^3}{27} - \frac{rr}{4}$ in the present case is $\frac{q^9}{27 \times 27 \times 27} - \frac{3q^6rr}{27 \times 27 \times 4} + \frac{3q^3r^4}{27 \times 16} - \frac{r^6}{64}$, which consists of the same terms as are contained in the cube of $\frac{rr}{4} - \frac{q^3}{27}$: but they are placed in a contrary order to that in which they stood in the former case; and the signs that are prefixed to them are contrary in every term to what they were before. And the same is true of all the other odd powers of these opposite differences of $\frac{rr}{4}$ and $\frac{q^3}{27}$.

34. It follows, therefore, that if ss be put for $\frac{q^3}{27} - \frac{rr}{4}$ in this latter case of the equation $x^3 - qx = r$, in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, the even powers of ss , to wit, s^4, s^8, s^{12}, s^{16} , &c. will represent, or be equal to, the same powers, products, and multiples of the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$ in the present case as they represented in the former case, when $\frac{rr}{4}$ was greater than $\frac{q^3}{27}$, and ss was made to stand for $\frac{rr}{4} - \frac{q^3}{27}$; and the several terms represented by the said even powers of ss will have the same signs

signs + and - prefixed to them respectively in this second case as in the first case. And it likewise follows that the odd powers of ss , to wit, ss , s^5 , s^{10} , s^{14} , &c. will also represent, or be equal to, the same powers, products, and multiples of the two original quantities $\frac{r}{4}$ and $\frac{q^3}{27}$, as in the former case; but the signs + and - prefixed to the several terms represented by the said odd powers of ss will be contrary to what they were before.

35. If therefore in this second case of the equation $x^3 - qx = r$, in which $\frac{r}{4}$ is less than $\frac{q^3}{27}$, we put $ss = \frac{q^3}{27} - \frac{r}{4}$, the series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \&c.$ will represent, or be equal to, a system of terms, derived from the two original quantities $\frac{r}{4}$ and $\frac{q^3}{27}$, that will be the very same in point of composition, that is, will be the very same powers, products, and multiples of $\frac{r}{4}$ and $\frac{q^3}{27}$, as the terms that were represented by it in the former case, in which $\frac{r}{4}$ was greater than $\frac{q^3}{27}$: but the terms so represented will not *all* have the same signs + and - prefixed to them as they had before; but those terms in the said system, which are represented by the terms of the series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{368s^6}{6561e^6} - \&c.$ which involve the even powers of ss , to wit, $\frac{20s^4}{243e^4}$, &c. will have the same signs prefixed

prefixed to them as they had before when ss stood for $\frac{rr}{4} - \frac{q^3}{27}$; and those terms of the said system which are represented by the terms of the said series which involve the odd powers of ss , to wit, $\frac{2ss}{9ee}$ and $\frac{308s^5}{6561e^6}$, &c. will have contrary signs to those they had before. Consequently, if we change the signs of those terms in the series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \&c.$ which involve the odd powers of ss , to wit, the terms $\frac{2ss}{9ee}$ and $\frac{308s^5}{6561e^6}$ &c. the new series thereby produced, to wit, $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.$ will represent, or be equal to, a system of terms which will not only be the very same in point of composition (or will be the same powers, products, and multiples of the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$), as those which were represented by the series $2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \&c.$ in the former case, but will also be connected with each other in exactly the same manner by the signs + and - : that is, by Art. 31. the said new series will represent, or be equal to, the following system of terms, to wit, $2 - \frac{a}{4} + \frac{aq^3}{27rr} - \frac{6}{16} + \frac{26q^3}{4 \times 27rr} - \frac{6q^6}{27 \times 27r^4} - \frac{y}{64} + \frac{37q^3}{16 \times 27rr} - \frac{37q^6}{4 \times 27 \times 27r^4} + \frac{7q^9}{27 \times 27 \times 27r^6} - \&c.$

But

But it has been shewn (in Art. 31.) that, if this system of terms be multiplied into $e^{\frac{1}{3}}$, or the cube-root of $\frac{r}{2}$, and the series thence produced be cubed, and also multiplied into q , and from its cube the product of its multiplication into q be subtracted, the remainder thereby obtained will be (nearly) equal to r . Therefore, if the series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.$ (which represents, or is equal to, the said system of terms, when $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, and ss is made $= \frac{q^3}{27} - \frac{rr}{4}$.) be multiplied by $e^{\frac{1}{3}}$, or the cube-root of $\frac{r}{2}$, and the series thence produced be cubed, and also multiplied into q , and from the cube of the said series the product of its multiplication into q be subtracted, it will follow that the remainder thereby obtained will be (nearly) equal to r ; that is, the product of the multiplication of $e^{\frac{1}{3}}$, or the cube-root of $\frac{r}{2}$, into the infinite series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.$ is equal to the value of x in the equation $x^3 - qx = r$ in the second case of it, when $\frac{rr}{4}$ is less than $\frac{q^3}{27}$. Q. E. I.

36. This series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.$ does not always converge, but only when ss is less than ee , or $\frac{q^3}{27} - \frac{rr}{4}$ is less than $\frac{rr}{4}$, or $\frac{q^3}{27}$ is less than $\frac{2rr}{4}$, or $\frac{rr}{4}$ is greater than half

half $\frac{q^3}{27}$, or than $\frac{q^3}{54}$, though less than $\frac{q^3}{27}$. And the nearer $\frac{rr}{4}$ approaches to $\frac{q^3}{27}$, the greater will be the swiftness with which this series will converge.

37. I will now add a few examples of the resolution of cubick equations of the aforesaid form, $x^3 - qx = r$, in the second case of those equations, in which r is less than $\frac{2q\sqrt[3]{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, by means of the infinite series $e^{\frac{x}{3}} \times \left[2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \text{&c.} \right]$ found in Art. 35. in order to confirm the truth of the reasonings by which that series was obtained.

E X A M P L E 1.

38. Let it be required to resolve the equation $x^3 - 50x = 120$ by means of the said infinite series.

Here q is = 50; r is = 120; $\frac{r}{2}$ or e , is = 60; $\frac{rr}{4}$, or ee , is = 3600; q^3 is = 125,000; and $\frac{q^3}{27}$ is = $\frac{125,000}{27} = 4629.629,629,629$, &c. which is greater than 3600, or $\frac{r}{2}$. Therefore this equation cannot be resolved by CARDAN'S rule, but may by the expression $e^{\frac{x}{3}} \times$ the series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \text{&c.}$ provided that series converges. Now, since $\frac{q^3}{27}$ is = 4629.629,629,629, &c.

and $\frac{rr}{4}$ is = 3600, we shall have $ss = \frac{q^3}{27} - \frac{rr}{4} = 4629.629$, 629, 629, &c. — 3600 = 1029.629, 629, 629, &c. which is considerably less than 3600, or ee ; and consequently the series will converge.

39. We shall therefore have $\frac{ss}{ee} = \frac{1029.629,629}{3600}$ &c.
 $= .286,00$; and $\frac{s^4}{e^4} = .081,796$; and $\frac{s^6}{e^6} = .023,393$; and
consequently $\frac{2ss}{9ee} = \frac{2 \times .286}{9} = \frac{.572}{9} = .063,55$; and $\frac{20s^4}{243e^4} =$
 $\frac{20 \times .081,796}{243} = \frac{1.635,92}{243} = .006,73$; and $\frac{308s^6}{6561e^6} = \frac{308 \times .023,393}{6561} =$
 $\frac{7.205,044}{6561} = .001,098$. Therefore $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6}$ is
 $= 2 + .063,55 - .006,73 + .001,09 = 2.064,64 - .006,73$
 $= 2.057,91$. And $e^{\frac{1}{3}}$, or $\sqrt[3]{e}$, is = $\sqrt[3]{60} = 3.914,867$.
Therefore $e^{\frac{1}{3}} \times$ the series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.$ is =
 $3.914,867 \times 2.057,91 = 8.0564$; that is, the root of the
proposed equation $x^3 - 50x = 120$ is 8.0564; which is
true in three places of figures, the error being in the
fourth place of figures, or third place of decimal frac-
tions, where the figure ought to be a 5 instead of a 6,
the more accurate value of x in that equation being
8.055,810,345,702, as may easily be found by Mr.
RAPHSON's method of approximation. But 8.0564,
the value of x found by the foregoing process, is suf-
ficiently near to its more accurate value 8.055,810, &c.

to

to shew the truth of the foregoing reasonings. Their difference is only $\frac{6}{10,000}$ parts of an unit, which is only the 13426^{th} part of $8.055,810$, &c. or the true value of x .

40. N. B. This equation $x^3 - 50x = 120$ expresseth the relation between the diameter of a circle and three chords in it that lie contiguous to each other, and together take up a semicircle, and form a trapezium of which the diameter of the circle is the fourth side. For if the three chords are called b , k and t , and the diameter of the circle is called x , the relation between them will be ex-

pressed by the cubick equation $x^3 - kk \left. \begin{array}{l} -bb \\ -tt \end{array} \right\} \times x = 2bkt$, which,

if the numbers 3, 4 and 5 are substituted instead of the letters b , k , and t , will become $x^3 - 50x = 120$. See Sir ISAAC NEWTON's Arithmetica Universalis, Edit. 2d. 1722, page 101.

E X A M P L E II.

41. Let it be required to find by means of the same series the root of the equation $x^3 - x = \frac{1}{3}$.

Now in this equation q is = 1, r is = $\frac{1}{3}$, $\frac{r}{2}$ is = $\frac{1}{6}$, $\frac{r^3}{27}$ is = $\frac{1}{27}$, or ee , is = $\frac{1}{36}$, and $\frac{q^3}{27}$ is = $\frac{1}{27}$, which is greater than $\frac{1}{36}$

or $\frac{r}{4}$. Therefore this equation cannot be resolved by CARDAN'S rule, but may by the series $e^{\frac{1}{3}} \times$
 $2 + - + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \text{&c.}$ in case that series is a converging one.

Now, since $\frac{q^3}{27}$ is $= \frac{1}{27}$, and $\frac{rr}{4}$ is $= \frac{1}{36}$, we shall have ss , or $\frac{q^3}{27} - \frac{rr}{4} = \frac{1}{27} - \frac{1}{36} = \frac{36 - 27}{27 \times 36} = \frac{9}{27 \times 36} = \frac{1}{3 \times 36}$, which is less than $\frac{1}{36}$, or ee , in the proportion of 1 to 3. Consequently the series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \text{&c.}$ and the series $e^{\frac{1}{3}} \times 2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \text{&c.}$ will converge. Therefore the equation $x^3 - x = \frac{1}{3}$ may be resolved by the means of it as follows.

42. Since ss is $= \frac{1}{3 \times 36}$, and $\frac{rr}{4}$, or ee , is $= \frac{1}{36}$, we shall have $\frac{ss}{ee} = \frac{1}{3} = .333,333$, and $\frac{s^4}{e^4} = \frac{1}{9} = .111,111$, and $\frac{s^6}{e^6} = \frac{1}{27} = .037,037$, and consequently $\frac{2ss}{9ee} = \frac{2 \times .333333}{9} = \frac{.666666}{9} = .074,074$, and $\frac{20s^4}{243e^4} = \frac{20 \times .111,111}{243} = \frac{2.222,222}{243} = .009,144$, and $\frac{308s^6}{6561e^6} = \frac{308 \times .037,037}{6561} = \frac{11.407,396}{6561} = .001,738$. Therefore $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \text{&c.}$ is $= 2 + .074,074 - .009,144 + .001,738 = 2.075,812 - .009,144 = 2.066,668$. And $\sqrt[3]{e}$ is $= \sqrt[3]{\frac{1}{6}} = \frac{1}{\sqrt[3]{6}} = \frac{1}{1.817,121}$. Therefore $e^{\frac{1}{3}} \times$ the series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \text{&c.}$ is $= \frac{1}{1.817,121} \times 2.066,668 =$

1.13733; that is, the root of the proposed equation $x^3 - x = \frac{1}{3}$ is 1.137,33; which is true to four places of figures, the error being in the fifth place of figures, or the fourth place of decimal fractions, where the figure ought to be an unit instead of a 3, the more accurate value of x being 1.137,158,164, which differs from the value of it here found by less than .00017, or $\frac{17}{100,000}$ parts of an unit, which is less than the 6689th part of 1.137,158,164, or the true value of x .

E X A M P L E III.

43. Let it be required to find the root of the equation

$$x^3 - 5x = 4.$$

Here q is = 5; r is = 4; $\frac{r}{2}$, or e , is = 2; $\frac{rr}{4}$, or ee , is = 4; q^3 is = 125, and $\frac{q^3}{27}$ is = $\frac{125}{27} = 4.629,629,629$, &c. which is greater than 4, or $\frac{rr}{4}$. Therefore this equation cannot be resolved by CARDAN's rule, but may by the infinite series $e^{\frac{1}{3}} \times \sqrt[3]{2 + \frac{ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.}$ in case that series is a converging one.

Now, since $\frac{q^3}{27}$ is 4.629,629,629, &c. and $\frac{rr}{4}$ is = 4, we shall have $\frac{q^3}{27} - \frac{rr}{4}$, or ss , = .629,629,629, &c. which is

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is less than 4, or ee , in the proportion of about 6 to 40, which is a pretty large proportion of minority, and much larger than the proportion of ss to ee in either of the former examples. Consequently the series $e^{\frac{1}{3}} \times$

$\boxed{2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \text{&c.}}$ will converge with a greater degree of swiftness than in either of those examples. Therefore the equation $x^3 - 5x = r$ may be resolved by it as follows.

44. Here $\frac{ss}{ee}$ is $= \frac{.629,629, \text{ &c.}}{4} = .157,407$; and consequently $\frac{s^4}{e^4}$ is $= .024,777$, and $\frac{s^6}{e^6}$ is $= .003,900$. Therefore $\frac{2ss}{9ee}$ is $= \frac{2 \times .157,407}{9} = \frac{.314,814}{9} = .034,979$, and $\frac{20s^4}{243e^4}$ is $= \frac{20 \times .024,777}{243} = \frac{.495,540}{243} = .002,039$, and $\frac{308s^6}{6561e^6}$ is $= \frac{308 \times .003,900}{6561} = \frac{1.201,200}{6561} = .000,182$, and consequently $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6}$ is $= 2 + .034,979 - .002,039 + .000,182 = 2.035,161 - .002,039 = 2.033,122$. And $e^{\frac{1}{3}}$, or $\sqrt[3]{e}$, is $= \sqrt[3]{2} = 1.259,921$. Therefore $e^{\frac{1}{3}} \times$ the series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \text{&c.}$ is $= 1.259,921 \times 2.033,122 = 2.561,573$; that is, the root of the proposed equation $x^3 - 5x = 4$ is 2.561,573; which is true to five places of figures, the error being in the sixth place of figures, or the fifth place of decimal fractions, where the figure ought to

be a 5 instead of a 7. For the accurate value of x in this equation is $\frac{1 + \sqrt{17}}{2}$, or $\frac{1 + 4.123,106}{2}$, or $\frac{5.123,106}{2}$, or 2.561,553; which differs from 2.561,573, or the value of x found by the foregoing series, by only $\frac{20}{1000,000\text{th}}$, or $\frac{2}{100,000\text{th}}$, parts of an unit, or less than the 128,000th part of 2.561,553, or the value of x itself; which is a great degree of exactness.

45. *Note.* That x , or the root of the equation $x^3 - 5x = r$, is accurately equal to $\frac{1 + \sqrt{17}}{2}$, will appear by substituting $\frac{1 + \sqrt{17}}{2}$ instead of x in the compound quantity $x^3 - 5x$, and observing that it will make that quantity become equal to 4. For, if x is $\frac{1 + \sqrt{17}}{2}$, we shall have $x^3 = \frac{1 + 3 \times \sqrt{17} + 3 \times 17 + 17 \times \sqrt{17}}{8} = \frac{5^2 + 20 \times \sqrt{17}}{8} = \frac{13 + 5\sqrt{17}}{2}$, and $5x = \frac{5 + 5\sqrt{17}}{2}$, and consequently $x^3 - 5x = \frac{13 + 5\sqrt{17}}{2} - \frac{5 - 5\sqrt{17}}{2} = \frac{8}{2} = 4$. Therefore $\frac{1 + \sqrt{17}}{2}$ is $= x$. Q. E. D.

46. These examples sufficiently prove that the expression $e^{\frac{x}{3}} \times$ the series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.$ (which we derived from the other series $e^{\frac{x}{3}} \times \left[2 - \frac{2ss}{9ee} - \frac{20s^4}{243e^4} - \frac{308s^6}{6561e^6} - \&c. \right]$ by the peculiar train of reasoning used in Art. 33, 34, and 35,) gives the true root of the cubick equation $x^3 - qx = r$

$qx = r$ in the second case of it, in which r is less than $\frac{2q\sqrt[3]{q}}{3\sqrt{3}}$, or $\frac{r}{4}$ is less than $\frac{q^3}{27}$, and which therefore cannot be resolved by CARDAN's rule.

I will, however, subjoin one more example to the same purpose; which shall be that of the equation $x^3 - 63x = 162$, which both DR. WALLIS and MR. DE MOIVRE have resolved by extracting what they call the impossible cube-roots of the impossible binomial quantities $81 + \sqrt{-2700}$ and $81 - \sqrt{-2700}$. Now this equation may be resolved by the foregoing expression $e^{\frac{1}{3}} \times$ the series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.$ in the manner following.

E X A M P L E 4.

47. Let it be required to find the root of the equation

$$x^3 - 63x = 162,$$

Here q is = 63; r is = 162; $\frac{r}{2}$, or e , is = 81; $\frac{r}{4}$, or ee , is = 6561; $\frac{q}{3}$ is = 21; and $\frac{q^3}{27}$ is = 9261, which is greater than 6561, or $\frac{r}{4}$. Therefore this equation cannot be resolved by CARDAN's rule, but may by the infinite series $e^{\frac{1}{3}} \times \left[2 + \frac{2ss}{9ee} - \frac{24s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c. \right]$ in case that series is a converging one.

Now,

Now, since $\frac{q^3}{27}$ is = 9261, and $\frac{r^r}{4}$ is = 6561, we shall have $\frac{q^3}{27} - \frac{r^r}{4}$, or ss , = 2700, which is less than 6561, or ee , in the proportion of 100 to 243. Consequently the series $2 + \frac{2ss}{9ee} - \frac{20^4}{243e^4} + \frac{308^6}{6461e^6} - \&c.$ and the product of that series multiplied into $e^{\frac{1}{3}}$, or the series $e^{\frac{1}{3}} \times 2 + \frac{2ss}{9ee} - \frac{20^4}{243e^4} + \frac{308^6}{6561e^6} - \&c.$ will converge. Therefore the equation $x^3 - 63x = 162$ may be resolved by it as follows.

48. Since ss is = 2700, and $\frac{r^r}{4}$, or ee , is = 6561, we shall have $\frac{ss}{ee} = \frac{2700}{6561} = \frac{100}{243} = .411,522$, and $\frac{s^4}{e^4} = .169,350$, and $\frac{s^6}{e^6} = .069,691$, and consequently $\frac{2ss}{9ee} = \frac{2 \times .411,522}{9} = \frac{823,044}{9} = .091,449$, and $\frac{20^4}{243e^4} = \frac{20 \times .169,350}{243} = \frac{3,387,0}{243} = .013,938$, and $\frac{308^6}{6561e^6} = \frac{308 \times .069,691}{6561} = \frac{21,464,828}{6561} = .003,271$. Therefore $2 + \frac{2ss}{9ee} - \frac{20^4}{243e^4} + \frac{308^6}{6561e^6} - \&c.$ is = $2 + .091,449, - .013,938, + .003,271 - \&c.$ = $2.094,720, - .013,938 - \&c.$ = $2.080,782, - \&c.$ And $e^{\frac{1}{3}}$, or $\sqrt[3]{e}$, is = $\sqrt[3]{81} = 4.326,749$. Therefore $e^{\frac{1}{3}} \times$ the series $2 + \frac{2ss}{9ee} - \frac{20^4}{243e^4} + \frac{308^6}{6561e^6} - \&c.$ is = $4.326,749 \times 2.080,782, - \&c.$ = $9.003,021 - \&c.$; that is, the root of the proposed equation $x^3 - 63x = 162$ is = $9.003,021, - \&c.$ or somewhat less than $9.003,021$; which is true to three places

of figures, the error being in the fourth place of figures, or the third place of decimal fractions, where there ought to be a cypher instead of a 3, because the accurate value of x in this equation is 9, as will appear upon trial : for, if x be taken = 9, we shall have $x^3=729$, and $63x=567$, and consequently $x^3-63x (=729-567)=162$.

S C H O L I U M.

49. This resolution of the equation $x^3-63x=162$ answers to Dr. WALLIS's resolution of it by extracting the cube-roots of the impossible binomial quantities $81+\sqrt{-2700}$ and $81-\sqrt{-2700}$, inasmuch as both resolutions are originally derived from CARDAN's rule. But the difference between them is, that the method here delivered is intelligible in every step of it, whereas Dr. WALLIS's method treats of impossible quantities, or quantities of which no clear idea can be formed, in the whole course of the process, though it concludes with a result that is intelligible, by means of the equality of the impossible members of the two ultimate quantities $\frac{9}{2} + \frac{1}{2}\sqrt{-3}$ and $\frac{9}{2} - \frac{1}{2}\sqrt{-3}$ (whose sum is equal to the value of x), and the contrariety of the signs + and -, which are prefixed to them. The doctor's method of finding $\frac{9}{2} + \frac{1}{2}\sqrt{-3}$ and $\frac{9}{2} - \frac{1}{2}\sqrt{-3}$ to be the cube-roots

of the impossible binomial quantities $81 + \sqrt{-2700}$ and $81 - \sqrt{-2700}$ is only tentative. But Mr. DE MOIVRE has given a *certain* method of finding the cube-roots of such quantities in all cases; but not without the trisection of an angle, or finding (by the help of a table of sines, or otherwise) the cosine of the third part of a circular arc whose cosine is given; by means of which trisection it is well known (independently of CARDAN's rule, or Mr. DE MOIVRE's process) that the second case of the cubick equation $x^3 - qx = r$ (in which $\frac{r}{4}$ is less than $\frac{q^3}{27}$, may be resolved. So that Mr. DE MOIVRE's method of doing this business, though more perfect than Dr. WALLIS's, does not seem to be of much use in the resolution of these equations. And both methods are equally liable to the objection above-mentioned, of exhibiting to our eyes, during the whole course of the processes, a parcel of algebraick quantities, of which our understandings cannot form any idea; though, by means of the ultimate exclusion of those quantities, the results become intelligible and true. It is by the introduction of such needless difficulties and mysteries into algebra (which, for the most part, take their rise from the supposition of the existence of negative quantities,

or quantities less than nothing, or of the possibility of subtracting a greater quantity from a lesser), that the otherwise clear and elegant science of algebra has been clouded and obscured, and rendered disgusting to numbers of men of a just taste for reasoning; who are apt to complain of it, and despise it, on that account. And, doubtless, they have too much reason to do so, and to say, in the words of the famous *Monsieur DES CARTES* in his dissertation *De Methodo*, page 11, *Algebram verò, ut solet doceri, animadverti certis regulis et numerandi formulis ita esse contentam, ut videatur potius ars quædam confusa, cuius usu ingenium quodammodo turbatur et obscuratur, quam scientia, quā excolatur et perspicacius reddatur.* If this complaint was just in *DES CARTES*'s time, there is certainly much more reason for it now.

50. The passage above alluded to in Dr. WALLIS's algebra, is in the 48th chapter, pages 179, 180, of the folio edition at London in 1685. And Mr. DE MOIVRE's method of extracting the cube-root of an impossible binomial quantity, as $81 + \sqrt{-2700}$, or $a + \sqrt{-b}$, is published in the appendix to the second volume of professor SAUNDERSON's algebra, pages 744, 745, 746, 747. It is very ingenious, and shews that author's great skill in the use and management of algebraick

gebraick quantities. See also on this subject CLAIRAUT's *Élémens d'Algébre*, Part V. Section 9. pages 286, 287, 288, and a paper of Monsieur NICOLE in the memoirs of the French Academy of Sciences for the year 1738, pages 99 and 100. See also MACLAURIN's algebra, Part I. the supplement to the 14th Chapter, pages 127, 128, 129, 130; and the Philosophical Transactions, N°. 451.

51. If any gentleman should be inclined to compute the series $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \&c.$ to more than four terms, he will find the first eight terms of it to be as follows, to wit, $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} - \frac{1870s^8}{59049e^8} + \frac{111,826s^{10}}{4,782,969e^{10}} -$

$$\frac{2,358,512s^{12}}{129,140,163e^{12}} + \frac{120,646,960s^{14}}{8,135,830,269e^{14}},$$
